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SELF-SIMILAR SOLUTIONS IN THE PROBLEM OF DIFFUSION OF A VORTEX IN A TEMPERATURE FIELD

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The author considers self-similar solutions of a system of nonlinear equations describing unsteady-state diffusion of a vortex with a heat source inside it with power dependences of the viscosity and the thermal conductivity on the temperature being included in the solution. Functions having a reasonable physical interpretation are singled out from the set of possible solutions.

1. In estimating the leveling of the vorticity of viscous liquids the following expression is ordinarily used for unsteady-state diffusion of the vortex [1]:

$$\omega = (\Gamma/2\pi\nu t)\exp(-r^2/4\nu t), \tag{1}$$

with the corresponding distribution of the circumferential velocity

$$V_\varphi = (\Gamma/2\pi r)\,[1 - \exp(-r^2/4\nu t)]. \tag{2}$$

However, a general theoretical analysis and experimental data show that real diffusion processes are more complicated than those predicted by Eqs. (1) and (2) and depend on many factors, including dynamic and thermal effects. In some cases Eqs. (1) and (2) cannot be considered satisfactory. In particular, when Eq. (2) is used, the total kinetic energy E_k of the vortex in the volume $0 < r < \infty$ turns out to be infinitely large at any moment of the diffusion process. It is evident that in real liquids the appearance of such vortices is doubtful. Moreover, the relations governing unsteady-state diffusion of vortices obtained using Eq. (1) are rather difficult to comprehend. The distribution $\omega(r, t)$ found in [1] by the superposition method shows that a vortex restricted at $t = 0$ $(r \le a)$ spreads instantaneously over the whole space $(0 < r < \infty)$ at any infinitesimal time $t > 0$.

At present there are two models free from this drawback that are characterized by a finite velocity of propagation of vortices. One of them is based on a generalized system of linear Onsager relations suggested by A. V. Luikov [2]. It leads to hyperbolic transfer equations and gives a defined front of disturbances that propagates with a finite velocity. The second model, which also leads to a finite velocity, is based on solution of nonlinear parabolic equations in which the transfer coefficients depend on the unknown functions (temperature, velocity, etc.) [3].

In what follows we will consider the second model as applied to the problem of the thermal effect upon the diffusion of a vortex with a heat source at the center. In the case of an incompressible viscous liquid a thermal effect on the velocity field can occur, provided that the viscosity coefficient depends on the temperature. Consequently, the equation for the vortex should be supplemented with a thermal equation in which the thermal conductivity can also depend on the temperature. As a result, a conjugate problem arises, which is described by the system of nonlinear equations

$$\rho\frac{\partial V_\varphi}{\partial t} = \frac{\partial}{\partial r}\left[\mu\left(\frac{\partial V_\varphi}{\partial r} - \frac{V_\varphi}{r}\right)\right] + \frac{2\mu}{r}\left(\frac{\partial V_\varphi}{\partial r} - \frac{V_\varphi}{r}\right),$$

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$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(rV_\varphi), \tag{3}$$

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r \frac{\partial T}{\partial r}\right) + Q. \tag{4}$$

Solutions of this system are sought in the self-similar statement with the boundary conditions $T \to T_\infty$ and $V_\varphi \to 0$ as $r \to \infty$. Following [4], the function $\vartheta = (T/T_\infty) - 1 \to 0$ at $r \to \infty$ is substituted for T and the power relations $\mu = \mu_0\vartheta^s$ and $\lambda = \lambda_\infty\vartheta^q$ are assumed. Then, characteristic scales of velocity, space, and time are introduced: V_0, $r_0 = \sqrt{\mu_0 t_0/\rho}$, and t_0. Accordingly, in what follows, the quantities V_φ, r, and t are dimensionless throughout. The time t_0 can be adopted on the basis of a prescribed value of the dimensional function $T(r, t)$, at a certain time at a chosen point r.

Formulated in this way, the system of equations (3) and (4) has many solutions. Their analysis will be carried out in the most generalized form to single out solutions that have a reasonable physical interpretation and are free from the drawbacks inherent in Eqs. (1) and (2).

2. The form of solutions of Eqs. (3) and (4) depends on the ratio of the exponents s and q. Only the case $s = q$ will be considered here. It holds for gases and corresponds to constant $\mathrm{Pr} = \mu c_p/\lambda$. A possible form of the solutions is their representation by the functions $V_\varphi = r^n\varphi(\eta)$ and $\vartheta = r^p f(\eta)$, where $\eta = rt^m$, $m = 1/(pq - 2)$, and n and p are arbitrary numbers. Along with this, a second representation $V_\varphi = rt^{(n-1)/(2-pq)}g(\eta)$ and $\vartheta = t^{p/(2-pq)}h(\eta)$ is possible, where $g(\eta) = \eta^{-1+n}\varphi(\eta)$ and $h(\eta) = \eta^p f(\eta)$. If in the second representation V_φ and ϑ are substituted into Eqs. (3) and (4), the following equations are found for determining $h(\eta)$ and $g(\eta)$:

$$(\eta h^q \dot{h})^{\cdot} + \frac{\mathrm{Pr}\,\eta}{2 - pq}(\eta\dot{h} - ph) + \mathscr{E}(\eta) = 0, \tag{5}$$

$$\eta\,(h^q \dot{g})^{\cdot} + 3h^q\dot{g} + \frac{\eta}{2 - pq}[(1 - n)\,g + \eta\dot{g}\,] = 0, \tag{6}$$

where $\mathscr{E}(\eta) = (Q\mathrm{Pr}t_0/\rho c_p T_\infty)rt^{(1-p(q+1))/(2-pq)}$; the dot denotes differentiation with respect to η.

Equations (7) and (8) are solved with the following boundary conditions: $h(0) = h_0$, $g(0) = g_0$, $h(\infty) = g(\infty) = 0$. In the energy estimation of the solutions, the enthalpy

$$E_t = 2\pi r_0^2 \rho c_p T_\infty \int_0^r \vartheta(\eta)\, r dr = 2\pi\mu_0 c_p t_0 T_\infty\, t^{\frac{p+2}{2-pq}} J_t(\eta), \tag{7}$$

where

$$J_t(\eta) = \int_0^\eta \eta^{p+1} f(\eta)\, d\eta \equiv \int_0^\eta \eta h(\eta)\, d\eta,$$

and the kinetic energy of the vortex

$$E_k = \pi\rho V_0^2 r_0^2 \int_0^r V_\varphi^2\, r dr = \pi\mu_0 t_0 V_0^2\, t^{\frac{2(n+1)}{2-pq}} J_k(\eta), \tag{8}$$

where

$$J_k = \int_0^\eta \eta^{2n+1}\varphi^2(\eta)\, d\eta \equiv \int_0^\eta g^2\eta^3 d\eta.$$

are considered.

It is evident that from the value of integrals (7) and (8) it is possible to evaluate the physical validity of solutions. In the case when the integrals are based on the whole volume, their values depend strongly on the asymptotic behavior of $h(\eta)$ and $g(\eta)$ at $\eta \to \infty$. This is seen from the general relations for $J_t(\infty)$ and $J_k(\infty)$ obtained by integration of Eq. (5) and Eq. (6) (preliminarily multiplied by $g\eta^2$) with respect to η between 0 and ∞:

$$(p+2)\,J_t(\infty) = J_Q\,(2-pq)/\mathrm{Pr} + [\eta^2 h(\eta) + \eta\dot{h}(\eta)\,(2-pq)/\mathrm{Pr}]_{\eta=\infty}, \tag{9}$$

where

$$J_Q = \int_0^\infty \eta\,\mathscr{E}(\eta)\,d\eta,$$

$$2J_k(\infty)\,(n+1)/(2-pq) = -2\int_0^\infty \eta^3 \dot{g}^2 h^q d\eta + [\eta^3 h^q (\dot{g}^2) + \eta^4 g^2/(2-pq)]_{\eta=\infty}. \tag{10}$$

The behavior of $g(\eta)$ and $h(\eta)$ depends, in turn, on p and n. The character of these relations will be considered preliminarily without thermal effects ($q = 0$).

3. First of all, it should be noted that the solutions $h(p, \eta)$ with $p > 0$ cannot satisfy the condition $h(p, \infty) = 0$. This can be shown by representing $\mathscr{E}(\eta) = \sum_{k=0}^{\infty} \mathscr{E}_k \eta^k$ and seeking a solution in the form of the series $h = h_0\,[1 + \sum_{k=1}^{\infty} h_k(p)\eta^k]$, where, after substitution of the series into Eq. (5), the coefficients $h_k(p)$ are determined by simple recurrence relations. Because of this the values $p > 0$ will not be considered further.

For $p < 0$ it is useful to seek solutions of Eq. (5) in the form of the series $h(p, \eta) = h_0[1 + \sum_{k=1}^{\infty} H_k(p)\xi^k]\exp(-\xi^2/2)$, where $\xi = \eta\sqrt{\mathrm{Pr}/2}$.

Representing the heat release function as $\mathscr{E}(\eta) = (\sum_{k=0}^{\infty} \mathscr{E}_k \xi^k)\exp(-\xi^2/2)$, it is easy to obtain an expression for the even coefficients of the series:

$$H_{2m} = D\begin{pmatrix} p+2m \\ p+2 \end{pmatrix} - \frac{\mathscr{E}_{2m-1}}{(2m)^2}\sum_{s=1}^{m-1}\frac{\mathscr{E}_{2s-1}}{(2s)^2}\,D\begin{pmatrix} p+2m \\ p+2+2s \end{pmatrix}; \quad m = 1, 2, \ldots \tag{11}$$

and for the odd coefficients of the series:

$$H_{2m+1} = -\frac{\mathscr{E}_{2m}}{(2m+1)^2} - \sum_{s=1}^{m}\frac{\mathscr{E}_{2s-2}}{(2s-1)^2}\,D\begin{pmatrix} p+2m+1 \\ p+2s+1 \end{pmatrix}; \quad m = 0, 1, 2, \ldots, \tag{12}$$

where

$$D\begin{pmatrix} p+\alpha \\ p+\beta \end{pmatrix} = \frac{(\beta+p)(\beta+p+2)(\beta+p+4)\ldots(\alpha+p)}{[\beta(\beta+2)(\beta+4)\ldots\alpha]^2}. \tag{13}$$

The solutions obtained satisfy the condition $h(p, \infty) = 0$ and describe the temperature variation $\vartheta = h(\xi)/t^{|p|/2}$ caused by a localized heat source present at the point $r = 0$ at the time $t = 0$. Without volume heat release ($\mathscr{E} = 0$), $H_{2m+1} = 0$ and H_{2m} vanishes for integral even values of $|p|$, starting from $m = -1 + |p|/2$. In this case the solutions $h(p, \xi)$ are elementary functions: in particular, for $|p| = 2, 4, 6, \ldots$ we have $h(-2, \xi) = h_0\exp(-\xi^2/2)$, $h(-4, \xi) = h_0(1-\xi^2/2)\exp(-\xi^2/2)$, $h(-6, \xi) = h_0(1 - \xi^2 + \xi^4/8)\exp(-\xi^2/2)$, etc. It should be noted that for $p = -2$ we have the well-known function of the heat source $\vartheta(\xi) = (h_0/t)\exp(-\xi^2/2)$.

For nonintegral values of $|p|$ the series for $h(p, \xi)$ is not reducible to elementary functions and therefore Eq. (5) was integrated numerically. Results of the integration are presented in Fig. 1a. It is noticeable that the value $p = -2$ is critical. For $0 > p \geq -2$ the functions $h(p, \xi) > 0$ over the whole range $0 < \xi < \infty$. For $p < -2$

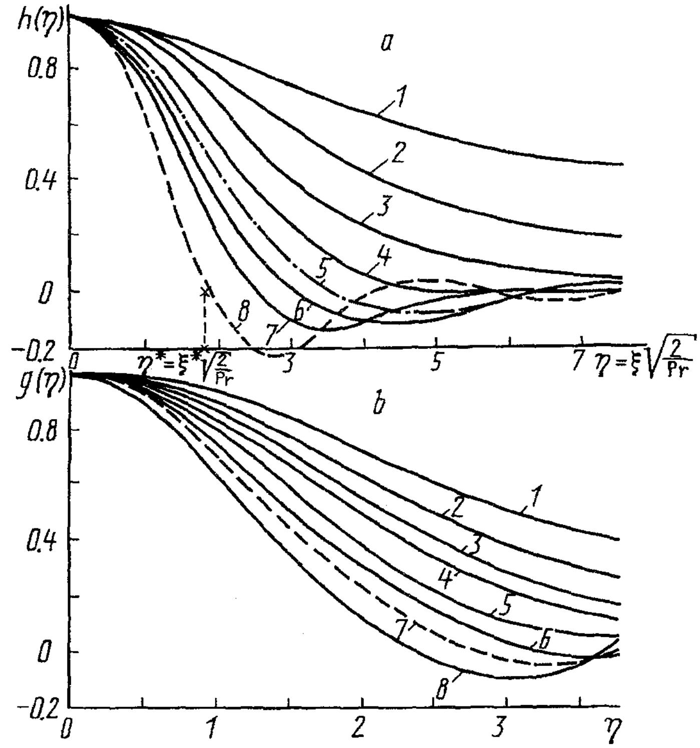

Fig. 1. Temperature functions $h(\eta)$ versus the parameter $p < 0$ (a): 1) $p = -0.5$; 2) -1; 3) -1.5, 4) -2; 5) -2.5; 6) -3; 7) -4; 8) -6; velocity functions versus the parameter $n < 0$ (b): 1) $n = -0.5$; 2) -1; 3) -1.5; 4) -2; 5) -2.5 6) -3.5; 7) -4.5; 8) -6; Pr = 0.7.

the function $h(p, \xi) > 0$ at $0 < \xi < \xi^*$, where ξ^* decreases as $|p|$ increases; at $\xi > \xi^*$ the function oscillates so that areas with negative and positive h and, accordingly, ϑ alternate.

Another distinction is that according to Eqs. (7) and (9) for $0 > p > -2$ the enthalpy $E_t(\infty) = \infty$ and for $0 > p > -2$, $E_t(\infty) = 0$; meanwhile, for $p = -2$, $E_t(\infty) = 4\pi t_0 \lambda T_\infty h_0$, i.e., the enthalpy is finite, nonzero, and constant in time. Proceeding from these values of $E_t(\infty)$, in estimating thermal disturbances from a localized source, it is expedient to use the function $h(-2, \xi)$, which is ordinarily done in practice. It is possible in principle to also use the functions $h(p, \xi)$ fort p < -2, but then it is difficult to interpret physically the appearance of regions of ξ with negative ϑ at any infinitesimal times $t > 0$.

4. In a similar way, we will consider a family of solutions $g(n, \eta)$ of Eq. (6) for $q = 0$. First of all, it should be noted that the solutions are invalid for $n > 0$ since $g(n, \infty) = \infty$. For $n < 0$, the solutions can be expressed in the form of the series $g(n, \eta) = g_0(1 + \sum_{k=1}^{\infty} g_k \eta^k) \exp(-\eta^2/4)$, where all the odd coefficients g_k are zero and the even coefficients are defined by the expressions

$$g_{2m}(n) = \frac{(n+3)(n+5)\dots(n+2m-1)}{2^{3m}(m!)^2(m+1)}; \quad k = 2m; \quad m = 1, 2, \dots \tag{14}$$

The series with these coefficients converges rapidly at any finite values of η. For integral odd $|n|$ the series is truncated, starting from $m = (|n| + 1)/2$, and the integrals in Eq. (6) are reduced to elementary functions. In particular, for $n = -3, -5$ ... we have $g(-3, \eta) = g_0 \exp(-\eta^2/4)$, $g(-5, \eta) = g_0(1-\eta^2/8) \exp(-\eta^2/4)$, etc. An exact solution can be also obtained for $n = -1$: $g(-1, \eta) = g_0[1-\exp(-\eta^2/4)]/\eta^2$; it is this solution that corresponds to distributions (1) and (2).

Equation (6) was solved numerically for nonintegral $|n|$. Results of the solution are given in Fig. 1b. The value $n = -3$ is critical for the family of curves obtained. For $0 > n \geq -3$, the function $g(n, \eta)$ decreases monotonically as $\eta \to \infty$, vanishing only at $\eta = \infty$, and for $n < -3$ regions of η with negative values appear in the functions $g(n, \eta)$. For $0 > n \geq -1$, the kinetic energy of the vortex is $E_k(\infty) = \infty$; for $-1 > n \geq -3$, $E_k(\infty)$ is finite and decreases monotonically with increase in $|n|$, in particular, it takes the value $E_k = 2\pi\mu_0 t_0 V_0^2 g_0^2/t^2$ for $n = -3$.

With these values of $E_k(\infty)$, reasoning as in the analysis of $h(p, \xi)$, physical evaluation of velocity disturbances from a vortex source concentrated initially at the point $r = 0$ can be made conveniently, using the solutions $g(n, \eta)$ for $-1 > n \geq -3$. Meanwhile, contrary to the generally accepted opinion, the function $g(-1, \eta)$, corresponding to Eqs. (1) and (2), should not be used in evaluations, since $E_k(\infty) = \infty$. The choice of the necessary value of n from the range $-1 > n \geq -3$ can be related to the boundary value of the vorticity at a point r, for example, $\omega = 2g_0/t^{(|n|+1/2)}$ at $r = 0$.

5. We will now consider the solution of Eqs. (5) and (6) in the case of an effect of the temperature field on the velocity field ($q \neq 0$). A simultaneous analytical solution of the equations can be found for $n = -3$, $p = -2$, and @ = 0:

$$h = h_0 \left[1 - \frac{\Pr q}{4(1+q)} \eta^2 \right]^{1/q}, \tag{15}$$

$$g = g_0 \left[1 - \frac{\Pr q}{4(1+q)} \eta^2 \right]^{1/(\Pr q h_0^q)}. \tag{16}$$

For the axisymmetric problem considered, solution (15) is similar to the relation obtained in [4] for the plane problem and is characterized by the front $\eta^* = 2\sqrt{(1+q)/(q\Pr)}$ bounding the heated area, which expands at a finite rate. In terms of the variables r and t, the front is determined by the relation

$$r^* = 2t^{2/(1+q)} \sqrt{(1+q)/(q\Pr)}. \tag{17}$$

It should be noted that the exponent in Eq. (15) coincides with that in [4], but the exponents in formula (17) and a similar expression in [4] are different: $1/(2+q)$ is replaced by $(2/(1+q)$. It follows from (17) that without heat interaction ($q = 0$) $r^* = \infty$ at any time.

Velocity function (16) is similar to $h(\eta)$ and indicates that under unsteady-state conditions the limited region of vorticity $0 < r < r^*$, expanding in accordance with (17), can exist. This is an entirely new physical result occurring for $q > 0$.

The integrals

$$J_t = \frac{2h_0}{\Pr} [1 - (1-\zeta)^{(1+q)/q}], \tag{18}$$

$$J_k = \frac{g_0^2 h_0^{2q} (1+q)^2}{(1 + \Pr q h_0^q)(2 + \Pr q h_0^q)} \left\{ 1 - (1-\zeta)^{2\alpha+1} [1 + (2\alpha+1)\zeta] \right\}, \tag{19}$$

where $0 \leq \zeta = \eta^2/\eta^{*2} \leq 1$ and $\alpha = h_0^q q \Pr$, entering into Eqs. (7) and (8) correspond to functions (15) and (16).

For these integrals the estimates hold at $\eta \to 0$:

$$J_t \to h_0 (\eta^2/2) \left(1 - \frac{1}{8} \frac{\Pr \eta^2}{1+q} \right) \quad \text{and} \quad J_k \to g_0^2 (\eta^2/4) \left[1 - \frac{\eta^2}{3h_0^q (1+q)} \right].$$

Using these estimates, it will be found that near the vortex center, the change in the enthalpy caused by q is equal to

$$\Delta E_t = E_t(q, \eta) - E_t(0, \eta) = (\pi/8) \mu_0 c_p t_0 T_\infty \Pr h_0 \frac{q}{1+q} \eta^4$$

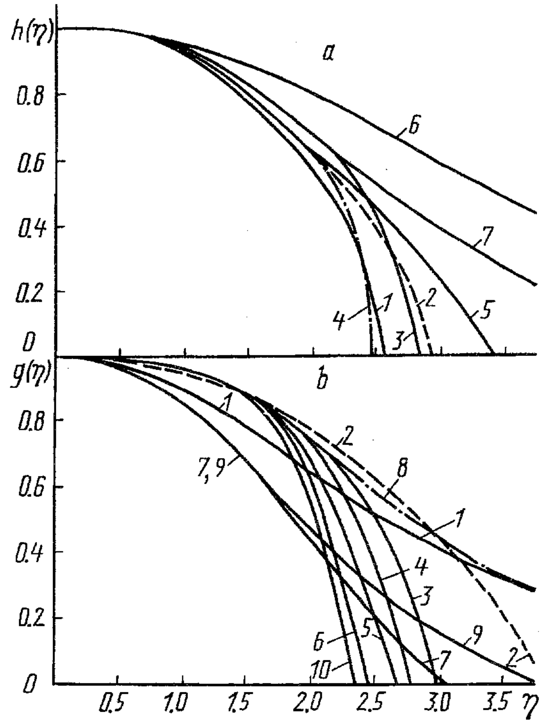

Fig. 2. Effect of the parameter $q > 0$ on the functions $h(\eta)$ for different $p < 0$ (a): 1) $p = 3$ and $q = 1$; 2) 2.5 and 1; 3) 2.5 and 2; 4) 2 and 2; 5) 2 and 1; 6) 1 and 1; 7) 1.5 and 0.5; on the functions $g(\eta)$ for different $p < 0$ and $n < 0$ (b): 1) $p = 2$, $n = 1$, and $q = 0$; 2) 2, 1, and 1; 3) 2, 1, and 1.5; 4) 2, 1, and 2; 5) 2, 1, and 3; 6) 2, 1, and 5; 7) 2, 4, and 1; 8) 1, 1, and 1; 9) 1, 3, and 1; 10) 4, 1, and 2 for $Q = 0$ and Pr = 0.7.

and the change in the kinetic energy is

$$\Delta E_{\mathrm{k}} = (\pi/4)\,\mu_0 t_0 V_0^2 t^{-2} \eta^4 \left[t^{2q/(1+q)} \left(1 - \frac{1}{3} \frac{h_0^{-q} \eta^2}{1+q} \right) - \left(1 - \frac{1}{3} \eta^2 \right) \right].$$

It is typical that $\Delta E_{\mathrm{t}} > 0$ and $\Delta E_{\mathrm{k}} > 0$, i.e., for $q > 0$ in the neighborhood of $r = 0$ both the thermal and kinetic energies increase.

Analytical solution of Eqs. (5) and (6) can also be found for $\mathscr{E} = \mathscr{E}_1 \eta$, which corresponds to the volume heat release $Q = \mathscr{E}_1 (t/t_0)^{-1} \rho c_p T_\infty / (\mathrm{Pr}\, t_0)$. Integration of Eq. (6) results in the relation

$$\frac{\eta^2}{4} + \int_{h_0}^{h} \frac{h^q dh}{\mathscr{E}_1 + \mathrm{Pr}\, h/(1+q)} = 0\,, \tag{20}$$

from which the boundary of the region is found, and beyond it $h = 0$:

$$\eta^{*2} = 4 \frac{(1+q)\, h_0^q}{\mathrm{Pr}} J_h(q, \mathscr{E}_1)\,, \quad \text{where} \quad J_h(q, \mathscr{E}_1) = \int_0^1 \frac{x^q dx}{x + \mathscr{E}_1 (1+q)/(\mathrm{Pr}\, h_0)}\,. \tag{21}$$

It follows from (20) that at $\mathscr{E}_1 > 0$, η^* is smaller than for $\mathscr{E}_1 = 0$, i.e., the boundary of the heated area is closer to $r = 0$. With account for Eq. (21), a sufficiently accurate approximation $h = h_0 (1 - \eta^2/\eta^{*2})^{1/q}$ is obtained from (20). Then, substituting this approximation into Eq. (6) an exact solution is found for the function $g(\eta)$:

$$g(\eta) = g_0 (1 - \eta^2/\eta^{*2})^{J_h/\mathrm{Pr} h_0^q}\,. \tag{22}$$

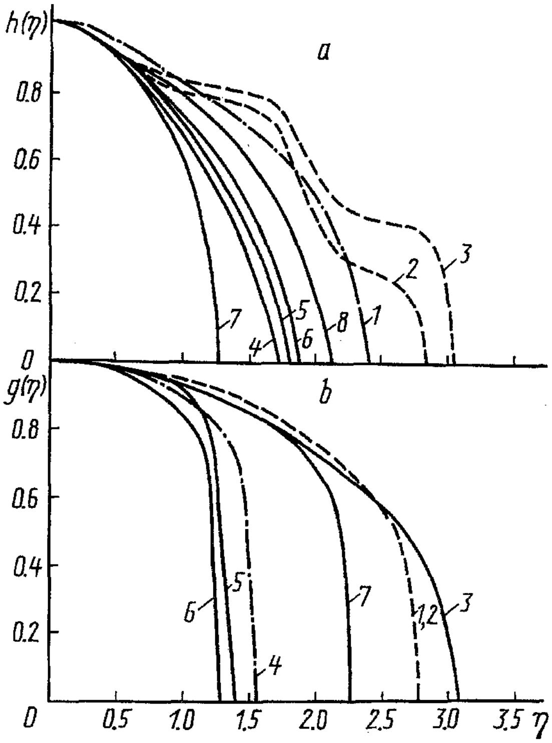

Fig. 3. Effect of the parameters of volume heat release Q on the temperature functions $h(\eta)$ for different $p < 0$ and $q > 0$ (a): 1) $|p| = 3$, $q = 1$, $A = 0$, $B = 1$, $\omega = 5$, $\beta = 1$; 2) 3, 1, 0, 1, 5, 0; 3) 2, 1, 0, 1, 5, 0; 4) 2, 0.5, 1, 0.5, 1, 1; 5) 2, 0.5, 1, 0.5, 4, 1; 6) 2, 0.5, 1, 0.5, 0.1; 7) 2, 1, 1, 0, 0, 0; 8) 2, 1, 1, 0, 0, 2; on the velocity functions $g(\eta)$ for different $q > 0$, $p < 0$, $n < 0$ (b): 1) $|p| = 3$, $q = 1$, $|n| = 2$, $A = 0$, $B = 1$, $\omega = 5$, $\beta = 1$; 2) 3, 1, 1, 9, 1, 5, 0; 3) 2, 1, 1, 0, 1, 5, 0; 4) 2, 0, 1, 1, 0, 0, 0; 5) 2, 1, 1, 1, 0, 0, 0; 6) 2, 1, 3,. 1, 0, 0, 0; 7) 2, 1, 1, 1, 0, 0, 2; Pr = 0.7.

Thus, for $Q \neq 0$, $q > 0$, there is a bounded region $(0 < \eta < \eta^*)$ with localized vorticity. Meanwhile, for $q = 0$ and $Q \neq 0$ this region is spread over the entire space. It should be emphasized that these relations are found for $p = -2$ and $n = -3$. However, solutions with the same properties can also be found for different values of p and n. To find them, Eqs. (5) and (6) were analyzed numerically.

6. Particular results of the analysis are as follows. For $q \neq 0$ not for all p does a bounded heated area occur. This is illustrated by curves 6 and 7 in Fig. 2a for $p = -1$ and 1.5, for which the asymptotic behavior $h(\eta) \to 0$ at $\eta \to \infty$ is monotonic. As regards the presence of localized areas $(0 < \eta < \eta^* < \infty)$, the value $p = -2$ is critical: for $0 > p > -2$ there are no such regions, and for $p < -2$ they exist for all $q > 0$. The effect of q itself decreases as $|p|$ decreases; in particular, for $p = -1$ and $q = 0$ the relation $h(0, \eta)$ (curve 2 in Fig. 1a) differs but slightly from the relation $h(1, \eta)$ for $q = 1$ (curve 6 in Fig. 2a).

At $\eta \to 0$ the effect of q consists in an increase in the values of $h(q, \eta)$, i.e., one of the results of the effect of q is an increase in the temperature in the neighborhood of the point $r = 0$. However, as η increases, $h(q, \eta)$ drops abruptly, starting from a certain value, and for $p \leq -2$ it vanishes at $\eta = \eta^*$.

The function $g(\eta)$ follows the changes in $h(\eta)$ in general. In particular, for $0 > p > -2$ the function $g(\eta)$ is characterized by the same asymptotic behavior as $h(\eta)$ (curves 8 and 9 in Fig. 2b). Similarly to $h(\eta)$, for all $p \leq -2$, there exists a bounded eddying zone with the same properties that are enumerated in Sec. 5 for the analytical solution for $p = -2$. This zone decreases as q, $|p|$, and $|n|$ increase (in Fig. 2b curves 1–6 for $p = -2$, $n = -1$ and various q; curve 7 for $p = -2$ and $n = -4$). A general analysis shows that the gradient $\dot{h}(\eta)$ has the greatest effect on changes in $g(\eta)$.

A change in the function $\mathscr{E}(\eta)$ is a means for affecting $h(\eta)$. In a numerical analysis of Eqs. (5) and (6), the function $\mathscr{E}(\eta) = (A + B \sin \omega\eta) \exp(-\beta\eta)$ was considered, which is physically sufficiently general and, in particular, includes the possibility of heat absorption ($\mathscr{E} < 0$ for $A = 0$ and $\eta > \eta_0 = \pi\omega$). In the last case a specific function $h(\eta)$ with inflections occurs (curves 2 and 3 for $p = -3, -2$, $\omega = 5$, $\eta_0 = 0.628$, and $\beta = 0$ in Fig. 3a). However, the inflections disappear for a value of the damping factor $\beta > 0$, for which the heat release Q decreases exponentially as the distance from $r = 0$ increases; ultimately, for $\beta = 1$ and $Q \neq 0$ the function $h(\eta)$ differs but slightly from the curve $h(\eta)$ obtained for $Q = 0$ (curves 1 in Figs. 2a and 3a). These characteristics of the behavior of $h(\eta)$ are also reflected in the functions $g(\eta)$ in Figs. 2b and 3b.

It should be noted that upon supply of heat ($Q > 0$), the vortex is relatively stabilized. This is manifested in the fact that at $0 < \eta < \eta^*$ the values of $g(\eta)$ are higher than those for $Q = 0$, for any value of n. The greatest excess occurs for $A = 1$ (curves 4–6 in Fig. 3b). It is noticeable that the points at which $g(\eta) = 0$ coincide with similar points η^* for $h(\eta)$. However, not in all cases do the functions $g(\eta)$ follow local changes in $h(\eta)$. In particular, curves 2 and 3 in Fig. 3b, corresponding to curves 2 and 3 in Fig. 3a, do not have inflection points. In general, the effect of $E_t(\eta)$ on $E_k(\eta)$ is integral; for all n, larger ΔE_k correspond to larger ΔE_t.

Thus, the nonlinearity of the system of equations (5) and (6) caused by $q > 0$, which is a parameter of the effect of the temperature field on the velocity field, generates localized unsteady-state thermal and eddying areas for $p \leq -2$ that expand at a finite rate. It is important that whereas for $q = 0$ only solutions for $p = -2$ and -1 $n \geq -3$ can be considered physically reasonable, for $q > 0$ the range of physically reasonable values of p and n expands: $p \leq -2$ and $n < -1$.

NOTATION

ω, vorticity; V_φ, circumferential velocity in the vortex; r, radial coordinate; t, time, ν and μ, kinematic and dynamic viscosities; λ, thermal conductivity; ρ, density of the medium; Pr, Prandtl number; Q and $\mathscr{E}$, volume heat release and its dimensionless value; n and p, exponents in power dependences of μ and λ on temperature; n, p, parameters in self-similar solutions; $\eta = rt^m$, reduced coordinate; $m = (pq-2)^{-1}$; $h(\eta)$, $g(\eta)$, temperature and velocity functions; E_t and E_k, enthalpy and kinetic energy of the vortex; A, B, ω, β, parameters in the heat release function; J_t and J_k, integrals in E_t and E_k; J_h, integral depending on the heat release; $D\begin{pmatrix}\alpha+p\\ \beta+p\end{pmatrix}$, parametric coefficients in the series; H_k, g_k, $\mathscr{E}_k$, coefficients in the series for h, g, and $\mathscr{E}$; η^*, reduced coordinates of the boundaries of the heated and eddying areas; $\Delta E_t = E_t(q)-E_t(0)$ and $\Delta E_k = E_k(q)-E_k(0)$, changes in the enthalpy and kinetic energy caused by the effect of q.

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